NASA TECHNICAL NOTE

NASA TN D-5878

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NASA TN D-5878

ANALYTICAL METHOD FOR STEADY STATE HEAT TRANSFER IN TWO-DIMENSIONAL POROUS MEDIA

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1970

1. Report No. NASA TN D-5878	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle ANALYTICAL METHOD FOR STEADY STATE HEAT TRANSFER IN TWO-DIMENSIONAL POROUS MEDIA		5. Report Date July 1970
		6. Performing Organization Code
7. Author(s) Robert Siegel and Marvin E. Goldstein		8. Performing Organization Report No. E-5526
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135		10. Work Unit No. 129-01
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Note
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

A general technique has been devised for obtaining exact solutions for the heat transfer behavior of two-dimensional porous cooled media. Fluid flows through the porous medium from a reservoir at constant pressure and temperature to a region which is at a lower pressure. For the type of flow involved the constant pressure surfaces are at constant velocity potential, so the porous region lies between parallel lines in a complex potential plane. The energy equation becomes separable when transformed to potential plane coordinates. The general solution for the temperature distribution can thus be obtained in the potential plane and then mapped into the physical plane.

17. Key Words (Suggested by Author(s)) Porous media; Transpiration cooling; Two-dimensional porous cooling	18. Distribution Statement Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	35	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

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SUMMARY

A general technique has been devised for obtaining exact solutions for the heat transfer behavior of a two-dimensional porous cooled medium. Fluid flows through the porous medium from a reservoir at constant pressure and temperature to a second reservoir at a lower pressure. For the type of flow involved, the surfaces of the porous region that are each at constant pressure are boundaries of constant velocity potential. This fact is used to map the porous region into a strip bounded by parallel potential lines in a complex potential plane. The energy equation, derived by assuming the local matrix and fluid temperatures are equal, is transformed into a separable equation when its independent variables are changed to the coordinates of the potential plane. This allows the general solution for the temperature distribution to be found in the potential plane. The solution is then mapped into the physical plane to yield the heat transfer characteristics of the porous region. An example problem of a porous wall having a step in thickness and a specified surface temperature or heat flux is worked out in detail.

INTRODUCTION

There are a number of flow and heat transfer applications that involve porous media. These include drainage of water in soil, the use of packed bed heaters, drying of materials, and transpiration cooling. In the last of these, with which this report is concerned, a coolant is made to flow through the porous material in order to protect this material when it is exposed to a high temperature environment. Possible applications for transpiration cooled materials are in rocket nozzles, gas turbine blades, and portions of vehicle surfaces in high speed flight.

Some of the early analyses of the heat transfer characteristics of porous cooled materials carried out around 1950 have been given in the heat conduction text by

Schneider (ref. 1). Some of the more recent papers are reference 2, which contains a long list of references, and reference 3. As can be seen from these references, the analyses of the heat flow within a porous cooled material have all been for one-dimensional geometries such as for a plane slab or radial flow through a tube wall. The objective of the present report is to present a method for obtaining analytical solutions in two-dimensional geometries.

It is evident that the configurations used for such devices as porous cooled turbine blades or rocket nozzles will deviate from a one-dimensional geometry. In some instances it may be possible to make a locally one-dimensional assumption in analyzing these geometries. However, the limits for making such an assumption can only be evaluated by examining solutions in more than one dimension. When the cross section of the material deviates considerably from a one-dimensional shape, the fluid flow in the porous matrix can follow complex paths, and two- or three-dimensional solutions are required. General two-dimensional solutions will be obtained here for either an arbitrary temperature variation or an arbitrary heat flux variation on the surface of the porous cooled medium. The relations that are found between surface temperature and heat flux would enable the solution for the heat transfer in the porous material to be coupled to the solutions to an external heat transfer problem such as a boundary layer flow.

The present analysis will utilize the fact that in many instances within the pores of a porous material the fluid and matrix material are in good thermal communication. As a result the local fluid temperature is the same as the local temperature of the matrix material. A single energy equation can then be written which is composed of two terms, one of which represents the energy carried by the flowing coolant and the other being the energy flow due to heat conduction in the matrix material.

The velocity that appears in the energy equation is a function of position in the medium, and for the slow viscous flow encountered in the pores of many porous media, the velocity is proportional to the local pressure gradient (Darcy's law). By suitably defining a dimensionless velocity and pressure, the dimensionless pressure is fixed at zero on the coolant reservoir side of the porous medium, and at unity on the high temperature side of the medium. Since the dimensionless velocity is proportional to the gradient of the dimensionless pressure, and since in addition, the velocity satisfies the continuity equation, the dimensionless pressure can be regarded as a velocity potential. Thus the region occupied by the porous material can be mapped into the region of a complex potential plane lying between two parallel potential lines at zero and unity which correspond to each of the constant pressure surfaces.

The method given here is based on transforming the energy equation so that its independent variables become the coordinates of the potential plane. In this plane the transformed energy equation is separable and the geometry is in the simple form of a unit strip. These facts make it possible to obtain a general solution to the energy equation. After the general solution has been obtained in the potential plane, conformal mapping is used to relate it to the coordinates of the physical plane.

The method described can be carried out for a variety of geometries and boundary conditions. The general solution is obtained here as a typical case for a two-dimensional porous region that is long in one of the coordinate directions and has a variable thickness in the other direction. One surface of the medium can have either an arbitrary specified temperature or heat flux.

To illustrate the application of the general solution, the specific situation of a step porous wall (i.e., a wall that changes in a step function fashion from one constant thickness to another) is considered. The solution is carried out to yield surface heat fluxes or temperatures corresponding to cases where one surface of the medium is either maintained at a specified uniform temperature, has a step function temperature variation, or has a specified uniform heat flux.

SYMBOLS

A	ratio of thick to thin dimensions of step porous wall
$C_1, C_2, C_3, C_4,$	integration constants
c_p	specific heat of fluid
$\tilde{E}$	transform defined by eq. (44)
$\tilde{\tilde{E}}$	function defined in eq. (65b)
F	function in specified temperature distribution eq. (6a)
G	function in specified heat flux distribution eq. (6b)
H	transform defined by eq. (39)
h_r	reference length in porous material
$\hat{i}, \hat{j}$	unit vectors in X- and Y-directions
k_m	thermal conductivity of porous region
L_S	dimensionless coordinate along boundary S, l_s/h_r
l_s	coordinate along boundary s such that $l_s = 0$ at $x = 0$
M	heat flux parameter, $(q_2 - q_1)/q_1$
N	temperature parameter, $(t_2 - t_1)/(t_1 - t_\infty)$
$\hat{n}$	outward normal vector
p	pressure
q	heat flux

S, S_o	bounding surfaces of porous region in dimensionless coordinate system
s, s_o	bounding surfaces of porous region
$\mathscr{S}$	boundary surface of arbitrary volume $\mathscr{V}$
T	dimensionless temperature defined in eqs. (10) and (11)
t	temperature
$\vec{U}$	dimensionless velocity $\frac{\mu}{\kappa} \frac{h_r}{p_o - p_s} \vec{u}$
$\vec{u}$	velocity
V	dimensionless velocity in Y direction
v	velocity in y direction
$\mathscr{V}$	arbitrary volume
W	complex potential $W = \psi + i\varphi$
X, Y	dimensionless coordinates, $\frac{x}{h_r}$ and $\frac{y}{h_r}$
x, y	rectangular coordinates
Z	dimensionless physical plane, $X + iY$
β	separation constant in solution for θ
Δ	quantity defined in eq. (11)
δ	Dirac delta function
ζ	intermediate mapping plane, $\zeta = \xi + i\eta$
η	imaginary part of ζ
θ	dependent variable defined by eq. (29)
κ	permeability of porous material
λ	parameter, $\frac{\rho C_p}{2k_m} \frac{\kappa(p_o - p_s)}{\mu}$
μ	fluid viscosity
ξ	real part of ζ
ρ	fluid density
Φ	function of φ in solution for θ
φ	potential, imaginary part of W, $(p_o - p)/(p_o - p_s)$
χ, ω	dummy variables of integration
Ψ	function of ψ in solution for θ

ψ real part of W

$\tilde{\nabla}$ dimensionless gradient, eq. (15)

Superscript:

at location of step in plate thickness

Subscripts:

o at the boundary s_o or S_o

s at the boundary S

∞ coolant reservoir

1, 2 values at large negative and large positive X (or x) on surface S (or s)

DERIVATION OF BASIC EQUATIONS

Let $\mathcal{V}$ be any volume within a porous medium with effective thermal conductivity k_m (based on the entire cross sectional area of the porous material) and permeability κ such that $\mathcal{V}$ is large with respect to the pore size and let $\mathcal{S}$ denote the surface of $\mathcal{V}$. Suppose that there is a fluid with constant density ρ, constant heat capacity C_p and constant viscosity μ which is flowing through $\mathcal{V}$. Assume that the thermal conductivity of the fluid is very small compared with k_m and that the pore size is so small that the fluid obeys Darcy's law. Let $\vec{u}$ denote the local Darcy velocity of the fluid (this is the velocity obtained by dividing the volume flow by the entire cross section of the porous material rather than the open area). If the thermal communication between the fluid and the matrix is sufficiently good, the local fluid temperature will be approximately equal to the matrix temperature in the immediate vicinity. We denote this common temperature by t. Finally, suppose that a steady state situation exists within $\mathcal{V}$. Then an overall energy balance applied to $\mathcal{V}$ shows that

$$\int_{\mathcal{S}} (\rho C_p t\vec{u} - k_m \nabla t) \cdot d\vec{\mathcal{S}} = 0$$

or applying the Divergence theorem

$$\int_{\mathcal{V}} \left[\rho C_p \nabla \cdot (t\vec{u}) - k_m \nabla^2 t\right] d\mathcal{V} = 0$$

If the changes in t and $\vec{u}$ are very small in distances on the order of the pore size, we can conclude from this (since $\mathcal{V}$ is arbitrary) that

$$k_m \nabla^2 t - \rho C_p \nabla \cdot (\vec{u}t) = 0 \tag{1}$$

Now the equation of continuity for the fluid is

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

This shows that equation (1) can be written as

$$k_m \nabla^2 t - \rho C_p \vec{u} \cdot \nabla t = 0 \tag{3}$$

Darcy's law for the fluid velocity is

$$\vec{u} = -\frac{\kappa}{\mu} \nabla p \tag{4}$$

where p is the local pressure of the fluid within the medium. Equations (2) to (4) are the three-dimensional generalizations of the classical one-dimensional porous cooling equations (see ref. 1).

GENERAL ANALYSIS OF TWO-DIMENSIONAL POROUS COOLED WALLS

The steps in the analysis involve (a) establishing the boundary conditions associated with the porous wall, (b) formulating these boundary conditions and the associated energy equation in dimensionless form, and (c) transforming the entire boundary-value problem into the potential plane and finding a general solution. The general solution applies to any arbitrary temperature or heat flux distribution along one surface of the porous material. Conformal mapping is used to relate the potential plane solution to the physical plane.

Boundary Conditions on Porous Region

Now consider the two-dimensional porous wall shown in figure 1. The lower surface of the wall whose unit outward-drawn normal is $\hat{n}_o$ is denoted by s_o, and the upper surface of the wall whose unit outward-drawn normal is $\hat{n}_s$ is denoted by s. We suppose that no changes occur in the direction perpendicular to the x, y-plane. Below the wall there is a reservoir which is maintained at constant pressure and temperature p_o and t_∞, respectively. The pressure in the fluid above the wall is constant and equal to p_s. We suppose that $p_o > p_s$. Then the fluid flows from the reservoir through the porous

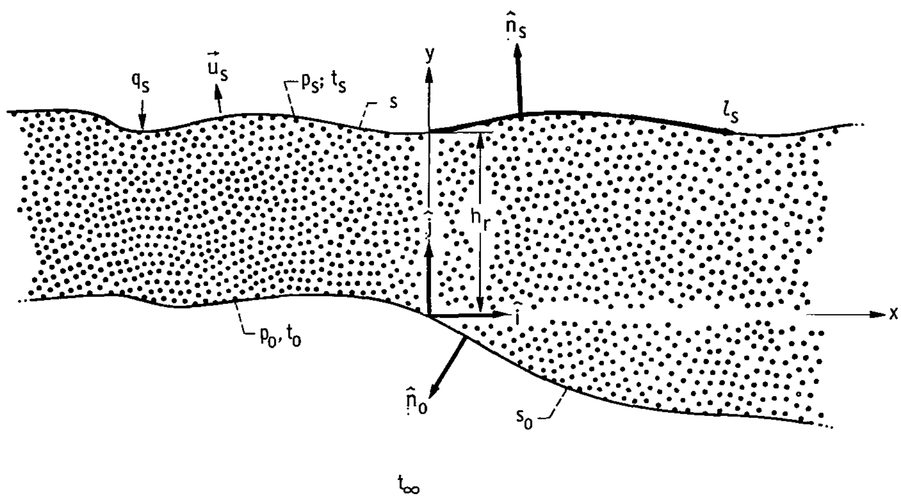


Figure 1. - Two-dimensional porous cooled wall.

wall and out through the top surface. Since p_o and p_s are both constant, the fluid velocity at the wall surfaces (s_o and s) will be in a direction perpendicular to these surfaces. The conditions on the upper surface of the wall will be chosen so that heat flows by conduction through the wall from the upper surface into the reservoir below at t_∞. We assume that the assumptions of the previous section apply within the wall and therefore that the flows of heat and mass within the wall are governed by equations (2) to (4).

Now as the fluid in the reservoir approaches the wall, its temperature will rise from the reservoir temperature t_∞ to the wall surface temperature t_o, which is an unknown variable along s_o. Since the thermal conductivity of the fluid is assumed to be much less than the thermal conductivity of the wall, for reasonable magnitudes of the fluid flow velocity the thickness of the fluid layer over which this temperature rise takes place is very small compared with the wall thickness. We can therefore assume that the fluid layer is locally one dimensional. Since the velocity is perpendicular to s_o, there is no flow along this thermal layer. Hence, applying an energy balance to the thermal layer shows that

$$\left.\begin{aligned} k_m\hat{n}_o \cdot \nabla t &= \rho C_p(t_o - t_\infty)\hat{n}_o \cdot \vec{u} \\ p &= p_o \end{aligned}\right\} \qquad \text{for } (x, y)\epsilon s_o \tag{5}$$

We shall consider two different types of boundary conditions for the upper surface s. First, we shall suppose that its surface temperature is specified so that it varies along the surface from a temperature t_1 at $l_s = -\infty$ to a temperature t_2 at $l_s = +\infty$. The l_s is the distance measured along the upper surface from the point where it intersects the y-axis. Thus in this case the boundary conditions on the upper surface of the wall are

$$\left.\begin{array}{l} t = t_s = t_1 + (t_2 - t_1)F\left(\dfrac{l_s}{h_r}\right) \\ p = p_s \end{array}\right\} \qquad \text{for } (x, y)\epsilon s \tag{6a}$$

where F is a given function which is equal to zero at $\left(\dfrac{l_s}{h_r}\right) = -\infty$ and equal to 1 at $\left(\dfrac{l_s}{h_r}\right) = +\infty$, and h_r is a reference length.

The second boundary condition which we shall consider is where the heat flux into the wall is specified along the upper surface so that it varies from a value of q_1 at $l_s = -\infty$ to a value of q_2 at $l_s = +\infty$. Under these conditions the boundary condition on the upper surface becomes

$$\left.\begin{array}{l} k_m \hat{n}_s \cdot \nabla t = q_s = q_1 + (q_2 - q_1)G\left(\dfrac{l_s}{h_r}\right) \\ p = p_s \end{array}\right\} \qquad \text{for } (x, y)\epsilon s \tag{6b}$$

where G is a given function which is equal to zero at $l_s/h_r = -\infty$ and equal to 1 at $l_s/h_r = +\infty$. Equations (2) to (4) together with the boundary conditions (5) and (6a) or alternatively together with the boundary conditions (5) and (6b) completely determine the solution to the heat transfer problems within the porous wall.

Dimensionless Form of Energy Equation and Boundary Conditions

It is now convenient to introduce the following dimensionless quantities:

$$\lambda \equiv \frac{\rho C_p}{2k_m} \frac{\kappa(p_o - p_s)}{\mu} \tag{7}$$

$$N = \frac{t_2 - t_1}{t_1 - t_\infty} \tag{8}$$

$$M = \frac{q_2 - q_1}{q_1} \tag{9}$$

$$\left.\begin{aligned} X &= x/h_r \\ Y &= y/h_r \\ L_s &= l_s/h_r \\ \varphi &= \frac{p_o - p}{p_o - p_s} \\ \vec{U} &= \frac{\mu}{\kappa}\frac{h_r}{(p_o - p_s)}\vec{u} \\ T &= \frac{t - t_\infty}{\Delta} \end{aligned}\right\} \tag{10}$$

where

$$\Delta \equiv \begin{cases} (t_1 - t_\infty) & \text{if boundary condition (6a) applies} \\ \dfrac{q_1 h_r}{k_m} & \text{if boundary condition (6b) applies} \end{cases} \tag{11}$$

Upon substituting these definitions into equations (2) to (4) and boundary conditions (5), (6a), and (6b), we obtain

$$\left.\begin{aligned} \tilde{\nabla}^2 T - 2\lambda\vec{U} \cdot \tilde{\nabla} T &= 0 \\ \vec{U} &= \tilde{\nabla}\varphi \\ \tilde{\nabla} \cdot \vec{U} &= 0 \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \hat{n}_o \cdot \tilde{\nabla} T &= 2\lambda\hat{n}_o \cdot \vec{U} T \\ \varphi &= 0 \end{aligned}\right\} \quad \text{for } (X, Y) \epsilon S_o \tag{13}$$

$$\left.\begin{aligned} T &= 1 + NF(L_s) \\ \varphi &= 1 \end{aligned}\right\} \quad \text{for } (X, Y) \epsilon S \tag{14a}$$

or

$$\left.\begin{aligned} \hat{n}_s \cdot \tilde{\nabla} T &= 1 + MG(L_s) \\ \varphi &= 1 \end{aligned}\right\} \qquad \text{for } (X, Y) \epsilon S \tag{14b}$$

where

$$\tilde{\nabla} \equiv \hat{i} \frac{\partial}{\partial X} + \hat{j} \frac{\partial}{\partial Y} \tag{15}$$

and the porous wall in dimensionless coordinates is shown in figure 2.

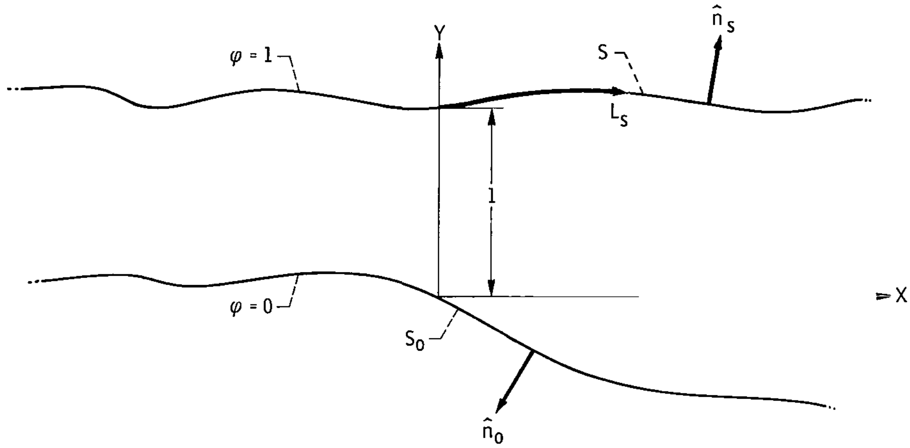


Figure 2. - Dimensionless physical plane (Z-plane).

The second equation (12) can be used to eliminate $\vec{U}$ in the other two equations (12). Thus

$$\tilde{\nabla}^2 T - 2\lambda \tilde{\nabla}\varphi \cdot \tilde{\nabla} T = 0 \tag{16}$$

and

$$\tilde{\nabla}^2 \varphi = 0 \tag{17}$$

Since φ is constant on S_o and S, it is clear that

$$\hat{n}_o = \frac{-\tilde{\nabla}\varphi}{|\tilde{\nabla}\varphi|} \qquad \text{for } (X, Y)\epsilon S_o$$

and

$$\hat{n}_S = \frac{\tilde{\nabla}\varphi}{|\tilde{\nabla}\varphi|} \qquad \text{for } (X, Y)\epsilon S$$

Using these results together with the second equation (12) in the boundary conditions (13) and (14b), we obtain

$$\left.\begin{aligned} \tilde{\nabla}\varphi \cdot \tilde{\nabla}T &= 2\lambda T|\tilde{\nabla}\varphi|^2 \\ \varphi &= 0 \end{aligned}\right\} \qquad \text{for } (X, Y)\epsilon S_o \tag{18}$$

and

$$\left.\begin{aligned} \tilde{\nabla}\varphi \cdot \tilde{\nabla}T &= |\tilde{\nabla}\varphi|\left[1 + MG(L_S)\right] \\ \varphi &= 1 \end{aligned}\right\} \qquad \text{for } (X, Y)\epsilon S \tag{19}$$

Transformation of Boundary Value Problem Into Potential Plane

Since equation (17) shows that φ satisfies Laplace's equation, there must exist a harmonic function ψ and an analytic function W of the complex variable

$$Z = X + iY \tag{20}$$

such that

$$W = \psi + i\varphi \tag{21}$$

Physically the change in ψ between any two points is proportional to the volume flow of liquid crossing any curve joining those two points. Hence ψ must vary between $-\infty$ and $+\infty$ as X varies between $-\infty$ and $+\infty$. Since

$$\mathcal{I}m W = 0 \qquad \text{for } Z\epsilon S_o$$

and

$$\mathcal{Im}\, W = 1 \qquad \text{for} \quad Z \in S$$

The mapping

$$Z \rightarrow W$$

transforms the physical plane (Z-plane) shown in figure 2 into the infinite strip shown in figure 3 in the W-plane. The boundaries S_0 and S in the physical plane are transformed into the lines $\varphi = 0$ and $\varphi = 1$, respectively, in the W-plane. The mapping $Z \rightarrow W$ can be found by using conformal mapping once the shape of the porous wall in the

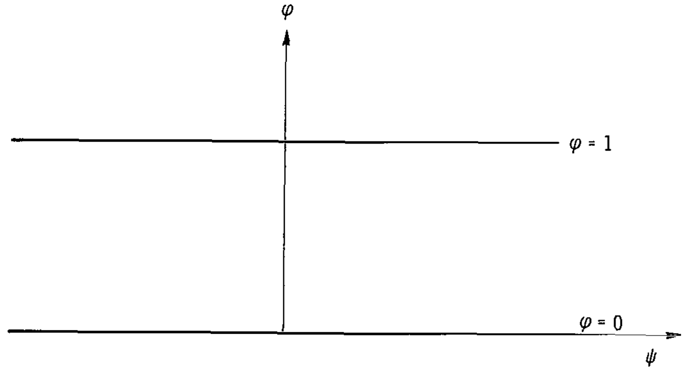


Figure 3. - Complex potential plane (W-plane).

physical plane is specified. This gives the solution to the boundary value problem for the flow. (An example for a specific wall geometry will be worked out subsequently.)

The boundary value problem for the temperature in the physical plane can be solved if we use the Boussinesq transform (ref. 4) to transform it into a boundary value problem in the W-plane. Thus, the independent variables X and Y in equation (16) and its boundary conditions will be changed to the variables φ and ψ, and the resulting boundary value problems will be solved in the infinite strip in the W-plane shown in figure 3. Once this solution, which gives T as a function of φ and ψ has been found, the mapping $W \rightarrow Z$ (which is completely determined once the geometry of the wall in the physical plane is specified) can be inverted to give Z as a function of φ and ψ. Thus T will be known parametrically (in terms of the parameters φ and ψ) as a function of X and Y. This will complete the solution to the problem.

To transform the energy equation (16) and boundary conditions into the potential plane, the following relations are used (ref. 5):

$$\tilde{\nabla}^2 T = \left(\frac{\partial^2 T}{\partial \psi^2} + \frac{\partial^2 T}{\partial \varphi^2}\right)\left|\frac{dW}{dZ}\right|^2 \tag{22}$$

and

$$|\tilde{\nabla}\varphi| = \left|\frac{dW}{dZ}\right| \tag{23}$$

Notice that

$$\frac{dW}{dZ} = \frac{dW}{dZ}\left(\frac{dZ}{dW}\frac{dW}{dZ}\right)^* = \left|\frac{dW}{dZ}\right|^2\left(\frac{dZ}{dW}\right)^*$$

Hence, upon taking real and imaginary parts, we find

$$\frac{\partial \varphi}{\partial X} = \left|\frac{dW}{dZ}\right|^2 \frac{\partial X}{\partial \varphi}$$

$$\frac{\partial \varphi}{\partial Y} = \left|\frac{dW}{dZ}\right|^2 \frac{\partial Y}{\partial \varphi}$$

This shows that

$$\tilde{\nabla}\varphi \cdot \tilde{\nabla}T = \left(\frac{\partial T}{\partial X}\frac{\partial X}{\partial \varphi} + \frac{\partial T}{\partial Y}\frac{\partial Y}{\partial \varphi}\right)\left|\frac{dW}{dZ}\right|^2 = \frac{\partial T}{\partial \varphi}\left|\frac{dW}{dZ}\right|^2 \tag{24}$$

Finally, notice that since φ is constant on S, the distance L_S along S is a function of ψ only. (This functional relation is known once the mapping $W \rightarrow Z$, which solves the flow problem, is known.) Thus

$$L_S(X, Y) = L_S(\psi) \qquad \text{for } (X, Y)\epsilon S \tag{25}$$

Now, using equations (22) to (25) in equation (16) and the boundary conditions (18), (14a), and (19) yields

$$\frac{\partial^2 T}{\partial \psi^2} + \frac{\partial^2 T}{\partial \varphi^2} - 2\lambda \frac{\partial T}{\partial \varphi} = 0 \tag{26}$$

$$\frac{\partial T}{\partial \varphi} - 2\lambda T = 0 \qquad \text{for} \quad \varphi = 0 \tag{27}$$

$$T = 1 + NF(L_s(\psi)) \qquad \text{for} \quad \varphi = 1 \tag{28a}$$

or

$$\frac{\partial T}{\partial \varphi} = \left|\frac{dZ}{dW}\right| \left[1 + MG(L_s(\psi))\right] \qquad \text{for} \quad \varphi = 1 \tag{28b}$$

Equation (26) can be simplified by introducing the new dependent variable θ defined by

$$\theta \equiv e^{\lambda(1-\varphi)}T = e^{\lambda(1-\varphi)} \frac{(t - t_\infty)}{\Delta} \tag{29}$$

In terms of this new variable equations (26) to (28b) become

$$\frac{\partial^2 \theta}{\partial \psi^2} + \frac{\partial^2 \theta}{\partial \varphi^2} - \lambda^2 \theta = 0 \tag{30}$$

$$\frac{\partial \theta}{\partial \varphi} = \lambda\theta \qquad \text{at} \quad \varphi = 0 \tag{31}$$

$$\theta = 1 + NF(L_s(\psi)) \qquad \text{at} \quad \varphi = 1 \tag{32a}$$

or

$$\frac{\partial \theta}{\partial \varphi} + \lambda\theta = \left|\frac{dZ}{dW}\right| \left[1 + MG(L_s(\psi))\right] \qquad \text{at} \quad \varphi = 1 \tag{32b}$$

Equation (30) together with the boundary conditions (31) and (32a) (or alternatively (31) and (32b)) constitute a boundary value problem for θ in the unit strip in the W-plane (shown in fig. 3) which completely determines θ (and therefore T) as a function of φ and ψ. Notice that the particular relation between Z and W (which depends only upon the geometry of the porous wall in the physical plane) enters only through the boundary condition (32a) (or alternatively (32b)) since L_s and $\left|\frac{dZ}{dW}\right|$ must be known as functions of ψ in order to completely determine these boundary conditions. It is possible to solve these boundary value problems for arbitrary L_S and $\left|\frac{dZ}{dW}\right|$ (as well as arbitrary F and G), and so the particular geometry and boundary conditions can be substituted into the general formulas once this general solution has been obtained.

General Solution of Boundary Value Problems in Potential Plane

The boundary value problems posed by equation (30) and the boundary conditions (31) and (32a) (or alternatively the boundary conditions (31) and (32b)) can easily be solved by the method of separation of variables. To this end we seek a solution to equation (30) of the form

$$\theta(\psi, \varphi) = \Psi(\psi)\Phi(\varphi) \tag{33}$$

Upon substituting this into equation (30), we find

$$\frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} = \lambda^2$$

This implies that there exists a constant β such that

$$\frac{\Psi''}{\Psi} = -\beta^2$$

and

$$\frac{\Phi''}{\Phi} = \lambda^2 + \beta^2$$

Hence,

$$\Phi = C_1 e^{\sqrt{\lambda^2+\beta^2}\,\varphi} + C_2 e^{-\sqrt{\lambda^2+\beta^2}\,\varphi} \tag{34}$$

$$\Psi = C_3 e^{i\beta\psi} + C_4 e^{-i\beta\psi} \tag{35}$$

where C_1 through C_4 are arbitrary constants of integration.

Upon substituting equations (33) to (35) into the boundary condition (31), we find that this boundary condition will be automatically satisfied if

$$C_1\sqrt{\lambda^2+\beta^2} - C_2\sqrt{\lambda^2+\beta^2} = \lambda(C_1 + C_2)$$

or

$$C_2 = C_1 \frac{\sqrt{\lambda^2 + \beta^2} - \lambda}{\sqrt{\lambda^2 + \beta^2} + \lambda}$$

Using this in equation (34) gives

$$\Phi = \frac{2C_1}{\sqrt{\lambda^2 + \beta^2} + \lambda}\left[\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right)\right]$$

Hence, the solution to both boundary value problems must be of the form

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\beta)\left[\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right)\right]e^{i\beta\psi}\,d\beta \tag{36}$$

where the function C of β can be determined so that either the boundary condition (32a) or the boundary condition (32b) is satisfied.

Solution to boundary value problem for specified temperature. - First suppose that the boundary condition (32a) applies. Put

$$\tilde{H}(\beta) \equiv C(\beta)\left[\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\right)\right]$$

Then equation (36) becomes

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{H}(\beta)\left[\frac{\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right)}{\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\right)}\right]e^{i\beta\psi}\,d\beta \tag{37}$$

Substituting this result into the boundary condition (32a) yields

$$1 + NF(L_s(\psi)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{H}(\beta)e^{i\beta\psi}\,d\beta$$

Upon inverting this Fourier transform, we find

$$\tilde{H}(\beta) = \int_{-\infty}^{\infty}\left[1 + NF(L_s(\psi))\right]e^{-i\beta\psi}\,d\psi$$

$$= \int_{-\infty}^{\infty} e^{-i\beta\psi}\,d\psi + N\int_{-\infty}^{\infty} F(L_s(\psi))e^{-i\beta\psi}\,d\psi$$

The first definite integral is equal to

$$\int_{-\infty}^{\infty} e^{-i\beta\psi}\, d\psi = 2\pi\delta(\beta)$$

where δ is the Dirac delta function. Hence

$$\widetilde{H}(\beta) = 2\pi\delta(\beta) + NH(\beta) \tag{38}$$

where

$$H(\beta) \equiv \int_{-\infty}^{\infty} F(L_s(\psi))e^{-i\beta\psi}\, d\psi \tag{39}$$

Substituting equation (38) into equation (37) yields

$$\theta = e^{\lambda(\varphi-1)} + \frac{N}{2\pi}\int_{-\infty}^{\infty} H(\beta)\left[\frac{\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\,\varphi\right)}{\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\right)}\right] e^{i\beta\psi}\, d\beta \tag{40}$$

Thus equation (40) with H defined in terms of the dimensionless temperature distribution F by equation (39) is the solution to the boundary value problem.

In this case it is also of interest to have an expression for the conduction heat flux q_s crossing into the surface S, that is,

$$q_s = k_m \hat{n}_s \cdot \nabla t \qquad \text{for } (x, y)\epsilon s$$

Therefore for $(X, Y)\epsilon S$, by use of equations (24) and (23),

$$\frac{q_s h_r}{k_m(t_1 - t_\infty)} = \hat{n}_s \cdot \widetilde{\nabla} T = \frac{\widetilde{\nabla}\varphi \cdot \widetilde{\nabla} T}{|\widetilde{\nabla}\varphi|} = \left|\frac{dW}{dZ}\right| \frac{\partial T}{\partial \varphi}$$

Hence, by use of the transformation (29) and the fact that $\varphi = 1$ on the boundary S we find

$$\frac{q_s h_r}{k_m(t_1 - t_\infty)} = \left|\frac{dW}{dZ}\right| \left(\frac{\partial\theta}{\partial\varphi} + \lambda\theta\right)\Bigg|_{\varphi=1} \tag{41}$$

Upon substituting equations (32a) and (40) into this expression, we obtain

$$\frac{q_s h_r}{k_m(t_1 - t_\infty)} = \left|\frac{dW}{dZ}\right|_{\varphi=1} \left\{ 2\lambda + \lambda N F(L_s(\psi)) \right.$$

$$\left. + \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{i\beta\psi} H(\beta)\sqrt{\lambda^2+\beta^2}\left[\frac{\lambda \cosh\left(\sqrt{\lambda^2+\beta^2}\right) + \sqrt{\lambda^2+\beta^2}\,\sinh\left(\sqrt{\lambda^2+\beta^2}\right)}{\lambda \sinh\left(\sqrt{\lambda^2+\beta^2}\right) + \sqrt{\lambda^2+\beta^2}\,\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}\right] d\beta \right\} \tag{42}$$

<u>Solution to boundary value problem for specified heat flux</u>. - Now suppose that the boundary condition (32b) applies. Put

$$\tilde{E}(\beta) \equiv C(\beta)\left[(2\lambda^2+\beta^2)\sinh\left(\sqrt{\lambda^2+\beta^2}\right) + 2\lambda\sqrt{\lambda^2+\beta^2}\,\cosh\left(\sqrt{\lambda^2+\beta^2}\right)\right]$$

Then equation (36) becomes

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\beta)\left[\frac{\lambda \sinh\left(\sqrt{\lambda^2+\beta^2}\,\varphi\right) + \sqrt{\lambda^2+\beta^2}\,\cosh\left(\sqrt{\lambda^2+\beta^2}\,\varphi\right)}{(2\lambda^2+\beta^2)\sinh\left(\sqrt{\lambda^2+\beta^2}\right) + 2\lambda\sqrt{\lambda^2+\beta^2}\,\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}\right] e^{i\beta\psi}\, d\beta \tag{43}$$

Substituting this result into the boundary condition (32b) yields

$$\left|\frac{dZ}{dW}\right|_{\varphi=1} \left[1 + MG(L_s(\psi))\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\beta) e^{i\beta\psi}\, d\beta$$

Upon inverting this Fourier transform, we obtain

$$\tilde{E}(\beta) = \int_{-\infty}^{\infty} \left|\frac{dZ}{dW}\right|_{\varphi=1} \left[1 + MG(L_s(\psi))\right] e^{-i\beta\psi}\, d\psi \tag{44}$$

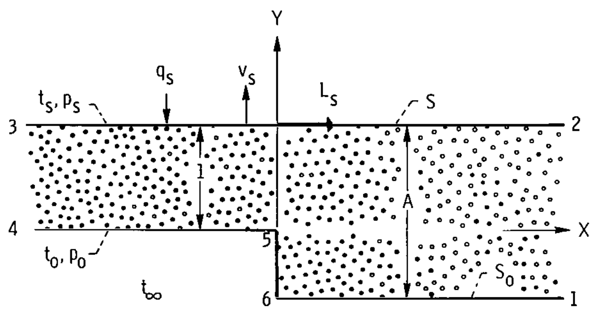
Thus equation (43) with $\tilde{E}$ defined in terms of the dimensionless heat flux distribution G and the reciprocal surface velocity $\left|\frac{dZ}{dW}\right|_{\varphi=1}$ by equation (44) is the solution to this boundary value problem. In particular the upper surface temperature distribution t_s (on S) is given by evaluating equations (29) and (43) at $\varphi = 1$ to obtain

$$\frac{(t_s - t_\infty)k_m}{q_s h_r} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{E}(\beta)$$

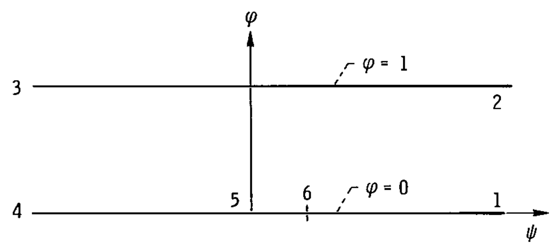
$$\left[\frac{\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\right) + \sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\right)}{(2\lambda^2 + \beta^2)\sinh\left(\sqrt{\lambda^2 + \beta^2}\right) + 2\lambda\sqrt{\lambda^2 + \beta^2}\cosh\left(\sqrt{\lambda^2 + \beta^2}\right)}\right]e^{i\beta\psi}\,d\beta \qquad (45)$$

SOLUTION FOR A STEP POROUS WALL

The general solution has now been obtained for either a specified arbitrary temperature or heat flux at one surface of the porous material. To illustrate the application of the solution, results will be obtained for the example of a step geometry as shown in figure 4(a). The thickness of the thin portion of the wall has been chosen as the reference dimension, so that all lengths are in units of this dimension.



(a) Dimensionless physical plane, Z = X + iY.



(b) Potential plane, W = ψ + iφ.

Figure 4. - Mapping of step porous wall into unit strip of potential plane.

Conformal Mapping Relations Between Z and W Planes

In the solutions the quantity $\left|\frac{dW}{dZ}\right|_{\varphi=1}$ appears along with the functional relation $L_s(\psi)$ relating positions along the surface S to positions along the boundary $\varphi = 1$ in the potential plane. These quantities can be found from the function $Z \rightarrow W$, which maps the region shown in figure 4(a) into the region shown in figure 4(b). This function can be obtained in terms of an intermediate variable ζ by utilizing the mapping given in reference 6. The ζ plane is shown in figure 5. The origin in the physical plane corresponds

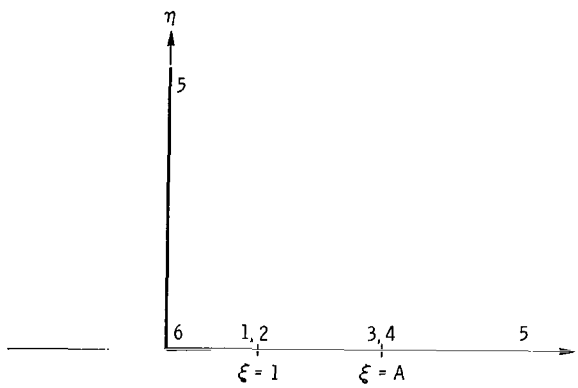


Figure 5. - Mapping of step porous wall into intermediate ζ -plane, $\zeta = \xi + i\eta$.

to $W = 0$ and $\zeta = \infty$. The mapping is given by

$$Z = \frac{1}{\pi}\left[A \log\left(\frac{\zeta + 1}{\zeta - 1}\right) - \log\left(\frac{\zeta + A}{\zeta - A}\right)\right] \tag{46}$$

and

$$\zeta = \xi + i\eta = \left(\frac{e^{\pi W} - A^2}{e^{\pi W} - 1}\right)^{1/2} \tag{47}$$

To obtain $dW/dZ\Big|_{\varphi=1}$ we use the relation

$$\frac{dW}{dZ} = \frac{dW}{d\zeta}\frac{d\zeta}{dZ}$$

The term $dZ/d\zeta$ is found from equation (46) to be

$$\frac{dZ}{d\zeta} = \frac{2A}{\pi} \frac{A^2 - 1}{(\zeta^2 - 1)(\zeta^2 - A^2)} \tag{48}$$

Upon solving equation (47) for W, we find

$$W = \frac{1}{\pi} \ln \frac{\zeta^2 - A^2}{\zeta^2 - 1}$$

Hence,

$$\frac{dW}{d\zeta} = \frac{2\zeta}{\pi} \frac{A^2 - 1}{(\zeta^2 - A^2)(\zeta^2 - 1)}$$

so that

$$\frac{dW}{dZ} = \frac{\zeta}{A} = \frac{1}{A}\left(\frac{e^{\pi W} - A^2}{e^{\pi W} - 1}\right)^{1/2} \tag{49}$$

At $\varphi = 1$, $e^{\pi W} = e^{\pi\psi}e^{i\pi} = -e^{\pi\psi}$. Therefore,

$$\left.\frac{dW}{dZ}\right|_{\varphi=1} = \frac{1}{A}\left(\frac{e^{\pi\psi} + A^2}{e^{\pi\psi} + 1}\right)^{1/2} \tag{50}$$

To obtain the relation between L_s and ψ along the boundary $\varphi = 1$, it is convenient to use ζ as an intermediate variable. The mapping defined by equation (46) transforms the porous region into the upper right quarter ζ-plane as shown in figure 5. The boundary $\varphi = 1$ maps onto the ξ-axis between $\xi = 1$ and $\xi = A$. The relation between L_s and ξ is found by letting $\zeta = \xi + i\eta$ in equation (46). Then taking the real part gives an expression for X as a function of ξ and η. Upon setting $\eta = 0$ in this expression, we obtain an equation for L_s as a function of ξ. The result is

$$L_s = \frac{1}{\pi}\left(A \log \frac{\xi + 1}{\xi - 1} - \log \frac{A + \xi}{A - \xi}\right) \qquad 1 < \xi < A \tag{51}$$

The relation between ξ and ψ along L_s is found from equation (47) by letting $\eta = 0$ and $\varphi = 1$ to obtain

$$\xi = \left(\frac{e^{\pi\psi} + A^2}{e^{\pi\psi} + 1}\right)^{1/2} \qquad -\infty < \psi < \infty \tag{52}$$

Thus equations (51) and (52) are parametric equations that relate L_s to ψ.

Exit Velocity From Porous Plate

It follows from equations (12) and (15) that for this example the velocity of the fluid leaving the porous wall is

$$V_s = \left|\frac{\partial\varphi}{\partial Y}\right|_{\varphi=1} \tag{53}$$

Since the fact that φ is constant along the upper boundary implies that $\partial\varphi/\partial X = 0$,

$$\left.\frac{dW}{dZ}\right|_{\varphi=1} = \left[-i\frac{\partial\psi}{\partial Y} + \frac{\partial\varphi}{\partial Y}\right]_{\varphi=1} = \left[i\frac{\partial\varphi}{\partial X} + \frac{\partial\varphi}{\partial Y}\right]_{\varphi=1} = \left.\frac{\partial\varphi}{\partial Y}\right|_{\varphi=1}$$

It therefore follows from equation (50) that

$$\frac{\mu}{\kappa}\frac{h_r}{(p_o - p_s)}v_s = V_s = \frac{1}{A}\left(\frac{e^{\pi\psi} + A^2}{e^{\pi\psi} + 1}\right)^{1/2} \tag{54}$$

Solution for a Step in Wall Temperature

In order to demonstrate how equation (42) can be applied, specific results will now be obtained for the example of a step function wall temperature variation along the boundary S. This solution also yields for zero step height the case of uniform wall temperature. In particular, let t_s change from t_1 to t_2 at an arbitrary location L_s' so that the boundary condition (14a) becomes

$$T = 1 + NF(L_s - L_s') \tag{55}$$

where F is the unit step function defined by

$$\begin{cases} F(X) = 1 & X > 0 \\ F(X) = 0 & X < 0 \end{cases}$$

The correspondence between L_s and ψ and L_s' and ψ' is obtained from equations (51) and (52). However, because of the special properties of the step function it is not necessary to relate L_s to ψ in order to evaluate $H(\beta)$. In fact, equation (39) becomes in this case

$$H(\beta) = \int_{-\infty}^{\infty} F(\psi - \psi')e^{-i\beta\psi}\, d\psi$$

or by letting $\chi = \psi - \psi'$

$$H(\beta) = e^{-i\beta\psi'} \int_{-\infty}^{\infty} F(\chi)e^{-i\beta\chi}\, d\chi$$

The evaluation of this integral is given in reference 7 as

$$H(\beta) = e^{-i\beta\psi'}\left[\pi\delta(\beta) + \frac{\text{P. V.}}{i\beta}\right] \tag{56}$$

where the notation P. V. indicates that the Cauchy principal value of the integral will be taken when integrating $1/i\beta$.

This expression for $H(\beta)$ can now be substituted into equation (42). Upon substituting the first term of equation (56) into the integral which appears in equation (42), we find that

$$\frac{N}{2\pi}\int_{-\infty}^{\infty} e^{i\beta(\psi-\psi')}\pi\delta(\beta)\sqrt{\lambda^2 + \beta^2}\left[\frac{\lambda \cosh\left(\sqrt{\lambda^2 + \beta^2}\right) + \sqrt{\lambda^2 + \beta^2}\,\sinh\left(\sqrt{\lambda^2 + \beta^2}\right)}{\lambda \sinh\left(\sqrt{\lambda^2 + \beta^2}\right) + \sqrt{\lambda^2 + \beta^2}\,\cosh\left(\sqrt{\lambda^2 + \beta^2}\right)}\right]d\beta = \frac{N}{2}\lambda \tag{57}$$

Substituting the remaining term of equation (56) into this integral yields the expression

$$\frac{N}{2\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{e^{i\beta(\psi-\psi')}}{i\beta}\sqrt{\lambda^2+\beta^2}\left[\frac{\lambda\cosh\left(\sqrt{\lambda^2+\beta^2}\right)+\sqrt{\lambda^2+\beta^2}\,\sinh\left(\sqrt{\lambda^2+\beta^2}\right)}{\lambda\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+\sqrt{\lambda^2+\beta^2}\,\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}\right]d\beta \tag{58}$$

As β goes to $+\infty$ and $-\infty$ the integrand approaches $\frac{1}{i}e^{i\beta(\psi-\psi')}$ and $-\frac{1}{i}e^{i\beta(\psi-\psi')}$, respectively so that the integral must be interpreted as the Fourier transform of a distribution. For this reason this expression is not suitable for numerical evaluation. However, in order to obtain an alternative expression for this integral which is suitable for this purpose, we need only add and subtract the term

$$\frac{N}{2\pi i}\int_{-\infty}^{\infty} E(\beta)e^{i\beta(\psi-\psi')}\,d\beta$$

where $E(\beta)$ is the sign function which is defined by

$$\begin{cases} E(\beta) = 1 & \beta > 0 \\ E(\beta) = -1 & \beta < 0 \end{cases}$$

It is found from a table of Fourier transforms of distributions that

$$\frac{1}{i}\int_{-\infty}^{\infty} E(\beta)e^{i\beta(\psi-\psi')}\,d\beta = \frac{2}{\psi-\psi'}$$

Hence the integral (58) can be written as

$$\frac{N}{2\pi}\frac{2}{(\psi-\psi')}+\frac{N}{2\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{i\beta(\psi-\psi')}}{i}\left\{\frac{\sqrt{\lambda^2+\beta^2}}{\beta}\left[\frac{\lambda\cosh\sqrt{\lambda^2+\beta^2}+\sqrt{\lambda^2+\beta^2}\,\sinh\sqrt{\lambda^2+\beta^2}}{\lambda\sinh\sqrt{\lambda^2+\beta^2}+\sqrt{\lambda^2+\beta^2}\,\cosh\sqrt{\lambda^2+\beta^2}}\right]-E(\beta)\right\}d\beta$$

$$=\frac{N}{2\pi}\frac{2}{(\psi-\psi')}+\frac{N}{2\pi}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}\frac{e^{i\beta(\psi-\psi')}}{i}\left\{\frac{\sqrt{\lambda^2+\beta^2}}{\beta}\left[\frac{\lambda\cosh\sqrt{\lambda^2+\beta^2}+\sqrt{\lambda^2+\beta^2}\,\sinh\sqrt{\lambda^2+\beta^2}}{\lambda\sinh\sqrt{\lambda^2+\beta^2}+\sqrt{\lambda^2+\beta^2}\,\cosh\sqrt{\lambda^2+\beta^2}}\right]-E(\beta)\right\}d\beta\right.$$

$$\left.+\int_{\epsilon}^{+\infty}\frac{e^{i\beta(\psi-\psi')}}{i}\left\{\frac{\sqrt{\lambda^2+\beta^2}}{\beta}\left[\frac{\lambda\cosh\sqrt{\lambda^2+\beta^2}+\sqrt{\lambda^2+\beta^2}\,\sinh\sqrt{\lambda^2+\beta^2}}{\lambda\sinh\sqrt{\lambda^2+\beta^2}+\sqrt{\lambda^2+\beta^2}\,\cosh\sqrt{\lambda^2+\beta^2}}\right]-E(\beta)\right\}d\beta\right]$$

Upon changing the variable of integration from β to $-\beta$ in the first integral, we find that the imaginary parts as well as the singularity at the origin cancel out, and we obtain

$$\frac{N}{2\pi}\frac{2}{(\psi - \psi')} + \frac{N}{\pi}\int_0^\infty \sin \beta(\psi - \psi') \times \left\{\frac{\sqrt{\lambda^2 + \beta^2}}{\beta}\left[\frac{\lambda \cosh\sqrt{\lambda^2 + \beta^2} + \sqrt{\lambda^2 + \beta^2}\sinh\sqrt{\lambda^2 + \beta^2}}{\lambda \sinh\sqrt{\lambda^2 + \beta^2} + \sqrt{\lambda^2 + \beta^2}\cosh\sqrt{\lambda^2 + \beta^2}}\right] - 1\right\} d\beta$$

or

$$\frac{N}{\pi(\psi - \psi')} + \frac{N}{\pi}\int_0^\infty \frac{(\lambda - \beta)\sqrt{\lambda^2 + \beta^2}\cosh\sqrt{\lambda^2 + \beta^2} + (\lambda^2 + \beta^2 - \lambda\beta)\sinh\sqrt{\lambda^2 + \beta^2}}{\lambda \sinh\sqrt{\lambda^2 + \beta^2} + \sqrt{\lambda^2 + \beta^2}\cosh\sqrt{\lambda^2 + \beta^2}} \times \frac{\sin[\beta(\psi - \psi')]}{\beta} d\beta \tag{59}$$

The integral in this expression is absolutely convergent, and hence, can easily be evaluated numerically. The results contained in relations (57), (59), and (50) show that equation (42) yields the following expression for the heat flux entering the boundary S:

$$\frac{q_s h_r}{k_m(t_1 - t_\infty)} = \frac{1}{A}\left(\frac{e^{\pi\psi} + A^2}{e^{\pi\psi} + 1}\right)^{1/2}\left\{2\lambda + N\lambda\left[\frac{1}{2} + F(L_s - L_s')\right] + \frac{N}{\pi(\psi - \psi')} + \frac{N}{\pi}\int_0^\infty \frac{(\lambda - \beta)\sqrt{\lambda^2 + \beta^2}\cosh\sqrt{\lambda^2 + \beta^2} + (\lambda^2 + \beta^2 - \lambda\beta)\sinh\sqrt{\lambda^2 + \beta^2}}{\lambda \sinh\sqrt{\lambda^2 + \beta^2} + \sqrt{\lambda^2 + \beta^2}\cosh\sqrt{\lambda^2 + \beta^2}} \times \frac{\sin[\beta(\psi - \psi')]}{\beta} d\beta\right\} \tag{60}$$

Solution for Uniform Heat Flux

In order to illustrate how equation (45) is applied, consider the case where a uniform heat flux is imposed along the surface S. The function G in equation (6b) is zero in this instance, and equation (50) gives an expression for $dW/dZ\Big|_{\varphi=1}$. Hence in this case the function $\widetilde{E}$ in equation (44) is given by

$$\frac{\widetilde{E}(\beta)}{A} = \int_{-\infty}^{\infty} \left(\frac{e^{\pi\psi} + 1}{e^{\pi\psi} + A^2}\right)^{1/2} e^{-i\beta\psi}\, d\psi \tag{61}$$

As $\psi \to \infty$ the integrand goes to $e^{-i\beta\psi}$ and as $\psi \to -\infty$ the integrand goes to $e^{-i\beta\psi}/A$, so that this integral exists only as the Fourier transform of a distribution and not as an ordinary integral. In order to separate out the singular parts of this integral, notice that equation (61) can be written as

$$\frac{\widetilde{E}(\beta)}{A} = \int_{-\infty}^{0} e^{-i\beta\psi}\left(\sqrt{\frac{e^{\pi\psi} + 1}{e^{\pi\psi} + A^2}} - \frac{1}{A}\right)d\psi + \int_{0}^{\infty} e^{-i\beta\psi}\left(\sqrt{\frac{e^{\pi\psi} + 1}{e^{\pi\psi} + A^2}} - 1\right)d\psi$$

$$+ \frac{1}{A}\int_{-\infty}^{\infty} e^{-i\beta\psi}\, d\psi + \left(1 - \frac{1}{A}\right)\int_{-\infty}^{\infty} F(\psi)e^{-i\beta\psi}\, d\psi$$

where the first two integrals are absolutely convergent and the last two integrals can be evaluated from a table of Fourier transforms to give

$$\frac{1}{A}\, 2\pi\delta(\beta) + \left(1 - \frac{1}{A}\right)\left[\pi\delta(\beta) + \frac{P.V.}{i\beta}\right]$$

Thus the following expression for $\widetilde{E}(\beta)$ is obtained

$$\frac{\widetilde{E}(\beta)}{A} = \pi\left(1 + \frac{1}{A}\right)\delta(\beta) + \left(1 - \frac{1}{A}\right)\frac{P.V.}{i\beta} + \int_{-\infty}^{0} e^{-i\beta\psi}\left(\sqrt{\frac{e^{\pi\psi} + 1}{e^{\pi\psi} + A^2}} - \frac{1}{A}\right)d\psi$$

$$+ \int_{0}^{\infty} e^{-i\beta\psi}\left(\sqrt{\frac{e^{\pi\psi} + 1}{e^{\pi\psi} + A^2}} - 1\right)d\psi \tag{62}$$

This is now inserted into equation (45). We shall consider the integrals resulting from the various terms of equation (62) individually. The integral resulting from the first term of equation (62) is

$$\frac{A}{2\pi}\pi\left(1+\frac{1}{A}\right)\int_{-\infty}^{\infty}\delta(\beta)\left[\frac{\lambda\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+\sqrt{\lambda^2+\beta^2}\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}{(2\lambda^2+\beta^2)\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+2\lambda\sqrt{\lambda^2+\beta^2}\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}\right]e^{i\beta\psi}\,d\beta$$

$$=\left(\frac{A}{2}+\frac{1}{2}\right)\frac{\lambda\sinh\lambda+\lambda\cosh\lambda}{2\lambda^2\sinh\lambda+2\lambda^2\cosh\lambda}=\frac{A+1}{4\lambda} \qquad (63)$$

The integral resulting from the second term of equation (62) is

$$\frac{(A-1)}{2\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\lambda\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+\sqrt{\lambda^2+\beta^2}\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}{(2\lambda^2+\beta^2)\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+2\lambda\sqrt{\lambda^2+\beta^2}\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}\frac{e^{i\beta\psi}}{i\beta}\,d\beta$$

$$=\frac{(A-1)}{\pi}\int_{0}^{\infty}\frac{\lambda\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+\sqrt{\lambda^2+\beta^2}\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}{(2\lambda^2+\beta^2)\sinh\left(\sqrt{\lambda^2+\beta^2}\right)+2\lambda\sqrt{\lambda^2+\beta^2}\cosh\left(\sqrt{\lambda^2+\beta^2}\right)}\frac{\sin\beta\psi}{\beta}\,d\beta \qquad (64)$$

This latter integral is absolutely convergent. Now consider the integrals resulting from inserting the last two terms of equation (62) into equation (45). These integrals are also found to be absolutely convergent. The exponential terms in the integrals can therefore be expressed in trigonometric form. After collecting real and imaginary parts, it is found that the imaginary part of the integrand is an odd function of β so that its integral from $\beta = -\infty$ to ∞ is zero. The integral of the remaining real part can be expressed as an integral from $\beta = 0$ to ∞ and is then combined with equations (63) and (64) to yield the following expression for the surface temperature:

$$\frac{(t_s - t_\infty)k_m}{q_s h_r} = \frac{A+1}{4\lambda}$$

$$+\frac{1}{\pi}\int_0^{\infty}\approx{E}(\beta,\psi)\left[\frac{\sqrt{\lambda^2+\beta^2}\cosh\sqrt{\lambda^2+\beta^2}+\lambda\sinh\sqrt{\lambda^2+\beta^2}}{2\lambda\sqrt{\lambda^2+\beta^2}\cosh\sqrt{\lambda^2+\beta^2}+(2\lambda^2+\beta^2)\sinh\sqrt{\lambda^2+\beta^2}}\right]d\beta \qquad (65a)$$

where

$$\tilde{\tilde{E}}(\beta, \psi) = (A - 1)\frac{\sin \beta\psi}{\beta} + \int_0^\infty \left\{\left(A\sqrt{\frac{1 + e^{\pi\omega}}{1 + A^2 e^{\pi\omega}}} - 1\right)\cos[(\psi + \omega)\beta] + A\left(\sqrt{\frac{e^{\pi\omega} + 1}{e^{\pi\omega} + A^2}} - 1\right)\cos[(\psi - \omega)\beta]\right\} d\omega \tag{65b}$$

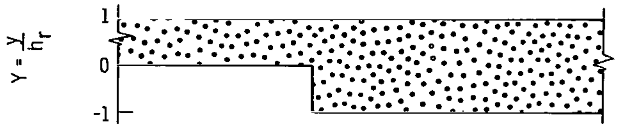
DISCUSSION

A general analytical method has been presented for determining the heat transfer characteristics of two-dimensional porous cooled media. This method is based on transforming the energy equation so that its independent variables are the coordinates in the potential plane for the fluid flow. The energy equation is separable in these coordinates, and hence a general solution can be obtained. A feature that aids in the solution is that the boundaries of the porous material are at constant pressure and as a consequence they map into parallel constant velocity potential lines in the complex potential plane. This provides a convenient region in which to solve the energy equation. The solution in the potential plane is then related to the physical plane by conformal mapping between the two regions.

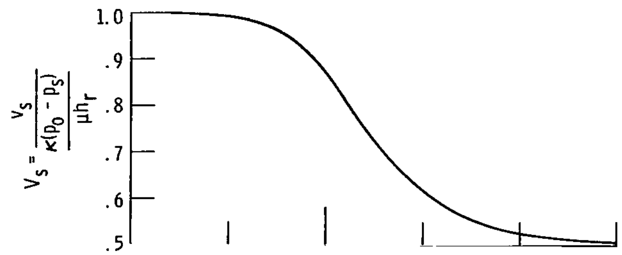
The following discussion concerns the application of the solution to the example of a porous wall with a step change in cross section. No attempt is made to conduct a parametric study to demonstrate the effect of the governing dimensionless parameters since this would only be of practical value if a particular engineering application were in mind.

Consider now the solution given by equations (51), (52), and (60). They give the dimensionless heat flux into the boundary S of the porous material parametrically as a function of position when the temperature distribution is a step function along that surface. The result contains the parameters λ, N, ψ', and A. The parameter A is the ratio of the thicknesses of the two regions of the porous wall as shown in figure 4(a). The only results which will be presented are for a value of A equal to 2. Hence, the geometry will always be as shown in figure 6(a). The heat flux is first computed as a function of ψ from equation (60). It is then expressed as a function of the coordinate L_S along the surface by use of the correspondence between L_S and ψ given by equations (51) and (52).

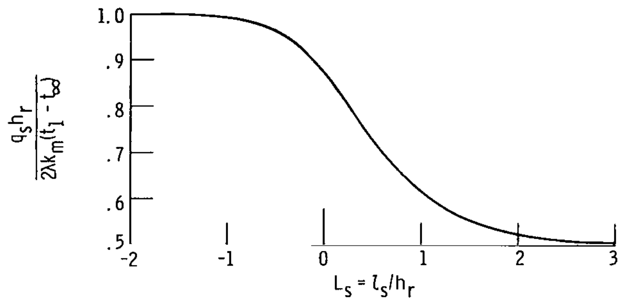
The dimensionless velocity leaving the upper surface is given by equation (54) and is shown in figure 6(b) for the porous plate with a step ratio A of 2. At the limits of large negative L_S and large positive L_S the velocity V_S goes, respectively, to 1 and 1/2, as



(a) Geometry of porous medium.



(b) Dimensionless exit velocity from upper surface of medium.



(c) Dimensionless heat flux at upper surface.

Figure 6. - Heat transfer to porous medium with upper surface at constant temperature, A = 2, N = 0.

would be expected by examining the one-dimensional solution given in reference 1. The extent of the two-dimensional region required for the velocity to change between the two limiting values is shown by the figure.

If the parameter $N = (t_2 - t_1)/(t_1 - t_\infty)$ is set equal to zero, then $t_2 = t_1$ and the surface S is at uniform temperature. With N = 0 the solution in equation (60) becomes

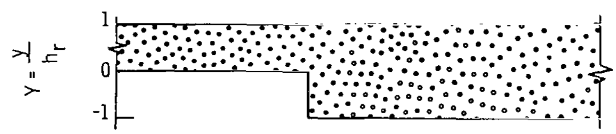
$$\frac{q_s h_r}{k_m(t_1 - t_\infty)} = \frac{1}{A}\left(\frac{e^{\pi\psi} + A^2}{e^{\pi\psi} + 1}\right)^{1/2} 2\lambda$$

so it follows from equation (54) that

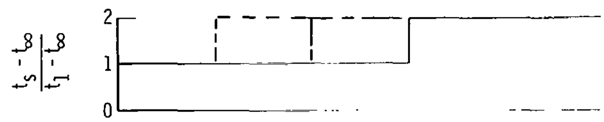
$$\frac{q_s h_r}{2\lambda k_m(t_1 - t_\infty)} = V_s$$

Thus for a given porous matrix when the surface temperature is uniform, the dimensionless heat flux into the boundary is directly proportional to the fluid exit velocity. The dimensionless heat flux is shown in figure 6(c). This curve is the same as the curve in figure 6(b).

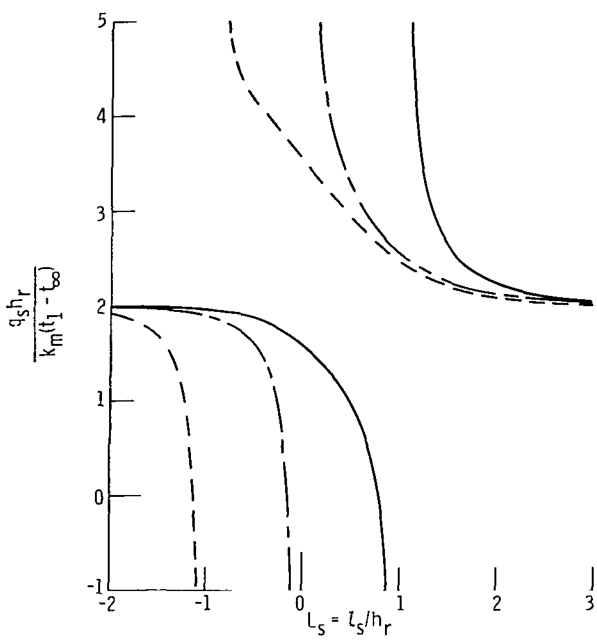
The results of the calculations for $N = 1$ and $\lambda = 1$ are shown in figure 7. In this case the surface temperature has the distribution shown in figure 7(b). The heat flux is computed for three different positions of the step corresponding to $L_s' = -1, 0, +1$. The values of ψ' which are the values of ψ corresponding to these values of L_s are found from equations (51) and (52), and then used in the solution equation (60). For large L_s the dimensionless heat flux is found to be the same as for large negative L_s, as shown



(a) Geometry of porous medium.



(b) Temperature distributions along upper surface.

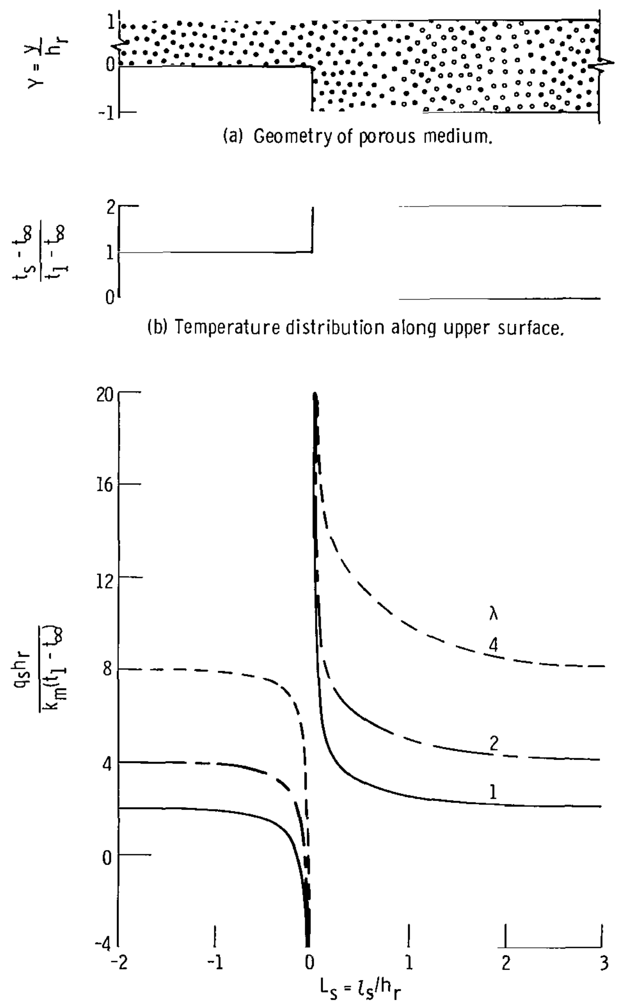


(c) Dimensionless heat flux at upper surface corresponding to temperature distributions in part (b).

Figure 7. - Heat transfer to surface for various locations of step in surface temperature, $A = 2$, $\lambda = 1$, $N = 1$.

by figure 7(c). This is the result of the compensating effects of having twice the temperature drop and half the exit velocity in the thick region of the wall as compared with the thin region. In the general case these two limiting values of q_s are not equal. In the vicinity of the temperature jump q_s changes discontinuously from a large negative value to a large positive value. This is a local heat conduction effect wherein the heat which is conducted into the porous wall at the high temperature side of the step flows into the low temperature region, where it is then extracted from the wall to maintain the surface temperature discontinuity.

Figure 8 shows the effect of varying the parameter $\lambda = \frac{\rho C_p}{2k_m} \frac{\kappa(p_o - p_s)}{\mu}$ while the



(c) Dimensionless heat flux at upper surface.

Figure 8. - Effect of λ on heat transfer to surface of step porous medium with step in surface temperature, A = 2, N = 1.

temperature distribution on the wall surface remains fixed as given in figure 8(b). The larger values of λ are associated for example with larger pressure differences $p_o - p_s$ and hence with larger flows. Thus to maintain a fixed surface temperature, the heat flux into the plate can be increased when the flow is increased to correspond to a larger value of λ.

Now consider the solution given by equations (51), (52), and (65) for the case of uniform heat input at the surface S. There are only two parameters involved, A and λ. Numerical results have been obtained for A = 2, and they are shown in figure 9. The exit velocity V_s is the same as in figure 6 and is repeated here for convenience in in-

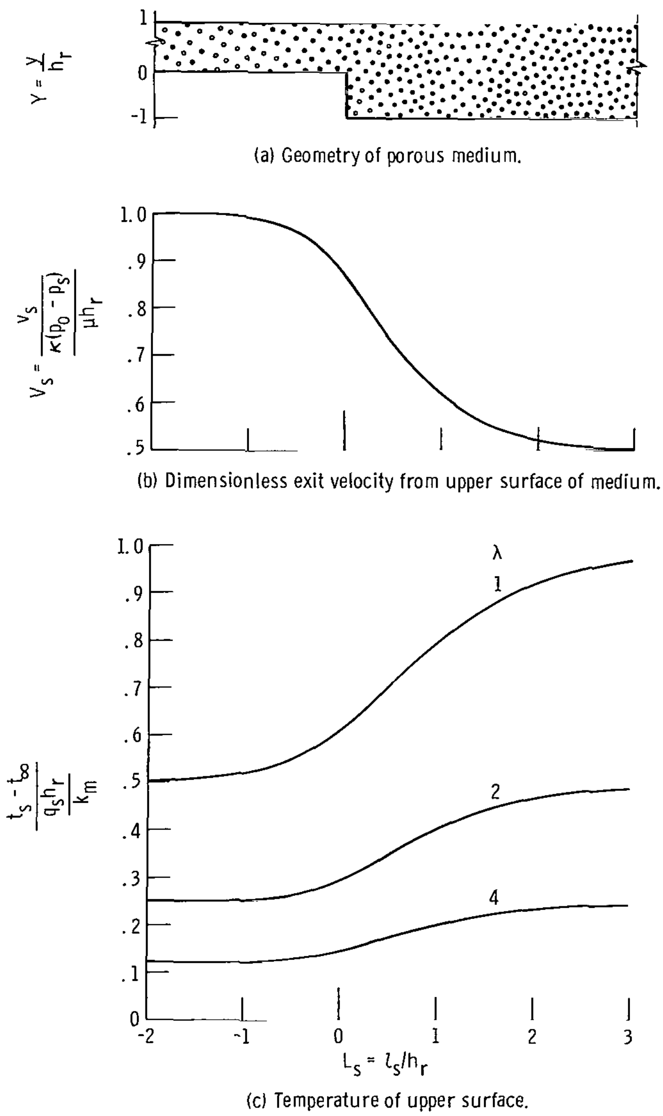


Figure 9. - Effect of λ on surface temperature for imposed uniform heat flux, A = 2.

terpreting the results. Since λ increases when the flow through the wall increases, the temperature difference $t_s - t_\infty$ between the surface and the reservoir decreases as λ is increased for a fixed q_s. Note that q_s has been combined into the dimensionless ordinate of figure 9(c) so that each curve can be used for any value of q_s. As would be expected, the regions of high surface temperature are associated with the regions of low exit velocity.

The preceding results serve to illustrate the type of heat transfer behavior to be expected in a two-dimensional porous configuration.

CONCLUSIONS

An analytical approach has been developed for obtaining the heat transfer behavior in a two-dimensional porous material. The method depends on the fact that the surfaces of the porous material at the inlet and outlet of the coolant are each at constant pressure. For the type of flow involved, the dimensionless pressure can be regarded as a velocity potential, and as a consequence the porous region maps into a simple strip in the complex potential plane. The energy equation was transformed into the potential plane, and a general solution was obtained. This solution can be applied to any wall geometry by finding the appropriate conformal map of the strip in the potential plane into the physical plane.

The analytical technique was applied for various imposed thermal conditions in the specific case of a step porous wall which is made up of two regions, each having a different uniform thickness.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, April 9, 1970,
129-01.

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NASA-Langley, 1970 —— 33 E-5526

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